Classification of Möbius maps

Monday, September 18, 2023 10:18 AM

Let us start looking for fixed points (can be $=\infty$ if $c=0$):

$$z = \frac{az+b}{cz+d} \iff \frac{cz^2+(d-a)z-b}{cz+d}=0 \qquad \text{Normalize: } ad-bc=1$$

Case 1: two fixed points: $(d-a)^2 \neq -4bc \overset{ad-bc=1}{\iff} (a+d)^2 \neq 4$.

Case 2: one fixed point: $(d-a)^2 = 4bc \iff (a+d) = \operatorname{Tr}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \pm 2$.
(parabolic).

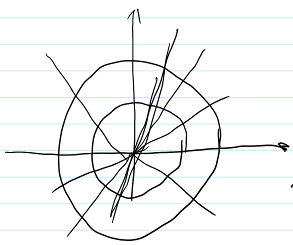
Case 1: Let z_1, z_2 be the fixed points.

Start with $z_1 = 0$, $z_2 = \infty$.

Then $c=0$, $b=0$. $T(z) = \frac{a}{d} z = a^2 z$
$ad=1$

Case 1a): $|a^2| = 1$, $a^2 \neq 1$. (not an identity)

$a^2 = e^{i\theta}$ $Tz = e^{i\theta} z$ – rotation.



Rotates the rays $l_\varphi\{\arg z = \varphi\}$ to $l_{\varphi+\theta}$

leaves circles $C_r = \{|z| = r\}$ invariant.

Elliptic.

$$\operatorname{Tr} T = a + \frac{1}{a} = e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} = 2\cos\frac{\theta}{2} \in (-2, 2)$$
$(\theta \in (0, 2\pi))$.

Case 1b) $a^2 > 0$, $a^2 \neq 1$. Hyperbolic.

$Tz = kz$, $k>0$, $k \neq 1$ – dilation.

Leaves the rays L_φ invariant.
Shifts C_r to C_{kr}.

$$\operatorname{Tr} T = \pm\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right) \in (-\infty, -2) \cup (2, \infty).$$

Case 1c $a^2 \notin \mathbb{R}_+ \cup \{|a^2| = 1\}$ – loxodromic – composition of elliptic and hyperbolic with the same fixed points. $\operatorname{Tr} T \notin \mathbb{R}$.

Arbitrary fixed points z_1, z_2

$$Sz := \frac{z - z_1}{z - z_2}: \quad Sz_1 = 0, \quad Sz_2 = \infty.$$

$\tilde{T} = STS^{-1}$: $\tilde{T}(\infty) = STS^{-1}(\infty) = ST(z_2) = S(z_2) = \infty$
$\tilde{T}(0) = 0$.

$\operatorname{Tr}\tilde{T} = \operatorname{Tr} T$ – by linear algebra.

Def T – elliptic/hyperbolic/loxodromic if so is $\tilde{T}$.

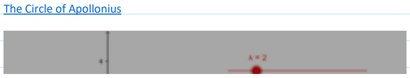
$\operatorname{Tr} T \in (-2, 2)$ – elliptic

$\operatorname{Tr} T \in (-\infty, -2) \cup (2, \infty)$

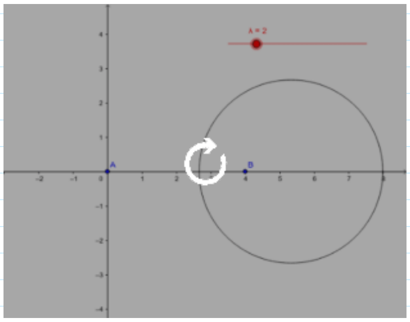
$T = S^{-1}\tilde{T}S$

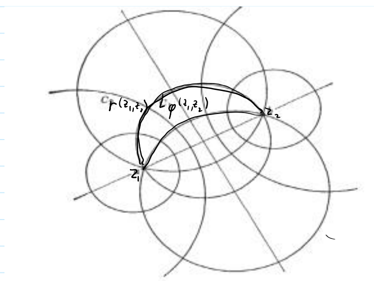
Geometry: $C_r(z_1, z_2) = \{|Sz| = r\} = \left\{\frac{|z - z_1|}{|z - z_2|} = r\right\}$ – circles of Apollonius.

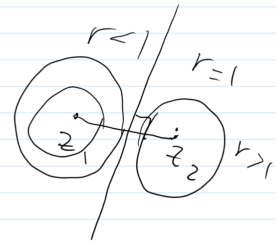
The Circle of Apollonius



[The Circle of Apollonius](#)







Steiner circles

$L_\varphi(z_1,z_2) = \{\arg(Sz)=\varphi\}$ - circular arcs from z_1 to z_2.

$L_\varphi(z_1,z_2) \perp C_r(z_1,z_2)$ - because S preserves angles.

Reflections: With respect to $C_r(z_1,z_2)$: preserves $L_\varphi(z_1,z_2)$, maps $C_s(z_1,z_2)$ to $C_{r^2/s}(z_1,z_2)$

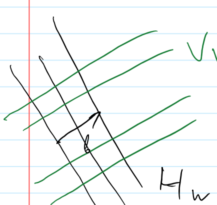
with respect to $L_\varphi(z_1,z_2)$: preserves $C_r(z_1,z_2)$, maps $L_\theta(z_1,z_2)$ to $L_{2\varphi-\theta}(z_1,z_2)$.

Proof. True for $z_1=0$, $z_2=\infty$. S preserves symmetry ■

Case 2 (parabolic) $\mathrm{Tr}\,T=\pm 2$. $T\neq id$.

z_0 - fixed point.

$z_0=\infty$. $Tz = z+b$, $b\neq 0$. (can be written as $T_{\begin{pmatrix}1&b\\0&1\end{pmatrix}}$ or $T_{\begin{pmatrix}-1&-b\\0&-1\end{pmatrix}}$)



$V_w^b = \{w+tb,\ t\in\mathbb{R}\}$ - vertical horocycles

$H_w^b = \{w+itb,\ t\in\mathbb{R}\}$ - horizontal

$V_w \perp H_{w'}$.

$TV_w^b = V_w^b$ $\quad TH_w^b = H_{w+b}^b$.

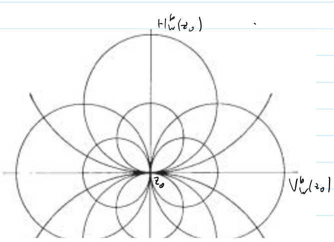
$z_0\neq\infty$. $\quad Sz := \dfrac{1}{z-z_0}$.

$\tilde T := STS^{-1}$; $\tilde T(\infty)=\infty$, no other fixed points

$\mathrm{Tr}\,T = \mathrm{Tr}\,\tilde T = \pm 2$. $\quad T = S^{-1}\tilde T S$

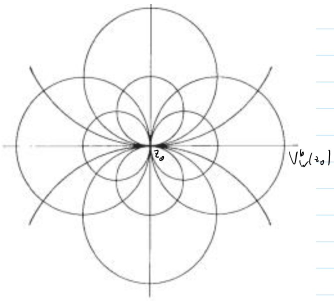
$V_w^b(z_0) = S^{-1}V_w^b$ - circles through z_0, tangent to line $S^{-1}b$.

$H_w^b(z_0) = S^{-1}H_w^b$ — ——— // ———— $iS^{-1}b$.



$TV_w^b(z_0) = V_w(z_0)$ $\quad TH_w^b(z_0) = H_{w+b}^b(z_0)$

Reflections are nice.



August Ferdinand Möbius

$T\,V_w^b(z_0) = V_w(z_0)$ $\quad T\,H_w^b(z_0) = H_{w+b}^b(z_0)$

Reflections are nice: